The Penrose transform for compactly supported cohomology.

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Abstract

Let the manifold X parametrise a family of compact complex submanifolds of the complex (or CR) manifold Z. Under mild conditions the Penrose transform typically provides isomorphisms between a cohomology group of a holomorphic vector bundle $V \to Z$ and the kernel of a differential operator between sections of vector bundles over X. When the spaces in question are homogeneous for a group G the Penrose transform provides an intertwining operator between representations.

In this paper we develop a Penrose transform for compactly supported cohomology on Z. We provide a number of examples where a compactly supported cohomology group is shown to be isomorphic to the cokernel of a differential operator between compactly supported sections of vector bundles over X. We consider also how the "Serre duality" pairing carries through the transform.

1 Introduction

Let the real manifold X parametrise a family of compact holomorphic submanifolds of a complex (or perhaps CR) manifold Z. If certain conditions are satisfied, the *Penrose transform* enables one to interpret Dolbeault cohomology with values in a holomorphic vector bundle on Z in terms of kernels and cokernels of differential operators on X. This procedure was developed in a representation-theoretic context by Schmid [S] and independently by Penrose [P] as part of his "Twistor Programme". (A cohomological interpretation of

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Penrose's work did not appear until [EPW], which considers the transform only in the holomorphic category.)

Let the set of points in X incident with each $z \in Z$ be contractible and of dimension d. Let $V \to Z$ be a holomorphic vector bundle. Then under conditions which appear below there is a spectral sequence

$$E_1^{p,q} = \Gamma(X, V_{p,q}) \Longrightarrow H^{p+q}(Z, V).$$

where the $V_{p,q}$ are certain complex vector bundles over X and the maps at the E_1 level are first order differential operators. Our main result is that in these circumstances one also has a spectral sequence

$$E_1^{p,q} = \Gamma_c(X, V_{p,q}) \Longrightarrow H_c^{p+q-d}(Z, V).$$

where the "c" subscripts refer to compactly supported sections and cohomology.

We also consider the "Serre duality" pairing between cohomology and compactly supported cohomology and the resulting pairing between the kernels and cokernels of differential operators arising from the transform.

2 Involutive structures and cohomology

We recall briefly the results about involutive structures and their cohomology that we will need. Details can be found in [BES]. We then define the compactly supported involutive cohomology. Our conventions are that $TM, \mathcal{E}_M^k$ will always refer to the *complexified* tangent bundle and the k-th exterior power of the complexified cotangent bundles of the real manifold M. In particular, $\mathcal{E}_M = \mathcal{E}^0$ refers to the trivial complex line bundle whose sections are complex-valued functions on M. We omit the "M" and write simply $\mathcal{E}^k$, etc, if there can be no confusion as to which manifold is intended.

Definition 2.1. *An <u>involutive structure</u> on the smooth manifold M is a complex sub-bundle $T^{0,1} \subset TM$ such that $[T^{0,1}, T^{0,1}] \subset T^{0,1}$ (meaning that the space of smooth sections is closed under Lie bracket). Define the vector bundle $\mathcal{E}^{1,0} \subseteq \mathcal{E}^1$ to be the annihilator of $T^{0,1}$ and define $\mathcal{E}^{0,1}$ by the exactness of*

$$0 \to \mathcal{E}^{1,0} \to \mathcal{E}^1 \to \mathcal{E}^{0,1} \to 0.$$

We write $\mathcal{E}^{p,q} = \wedge^p \mathcal{E}^{1,0} \otimes \wedge^q \mathcal{E}^{0,1}$.

Definition 2.2. *A complex vector bundle $V \to M$ is compatible with the involutive structure $\mathcal{E}$ (or $\mathcal{E}$-compatible) if there is defined a linear operator*

$$\bar{\partial} : \Gamma(M, V) \to \Gamma(M, V \otimes \mathcal{E}^{0,1})$$

such that

$$\bar{\partial}(fs) = f\bar{\partial}(s) + (\bar{\partial}f)s, \quad \forall f \in \mathcal{E}(M), s \in \Gamma(M, V)$$

and such that the extension to

$$\bar{\partial} : \Gamma(M, V \otimes \mathcal{E}^{0,q}) \to \Gamma(M, V \otimes \mathcal{E}^{0,q+1})$$

satisfies $\bar{\partial}^2 = 0$.

Definition 2.3. *Given an $\mathcal{E}$-compatible vector bundle $V \to M$ define the involutive cohomology $H^*_{\mathcal{E}}(M, V)$ to be the cohomology of the complex*

$$\Gamma(M, V) \xrightarrow{\bar{\partial}} \Gamma(M, \mathcal{E}^{0,1} \otimes V) \xrightarrow{\bar{\partial}} \Gamma(M, \mathcal{E}^{0,2} \otimes V) \xrightarrow{\bar{\partial}} \cdots$$

Writing $\Gamma_c(M, V)$ for the compactly supported smooth sections of V, etc, we define the compactly supported involutive cohomology $H^*_{\mathcal{E},c}(M, V)$ to be the cohomology of the complex

$$\Gamma_c(M, V) \xrightarrow{\bar{\partial}} \Gamma_c(M, \mathcal{E}^{0,1} \otimes V) \xrightarrow{\bar{\partial}} \Gamma_c(M, \mathcal{E}^{0,2} \otimes V) \xrightarrow{\bar{\partial}} \cdots$$

We will be using the following examples of involutive structures:

1. An involutive structure $T^{0,1} \subset TM$ is a *complex structure* iff $TM = T^{0,1} \oplus \overline{T^{0,1}}$. In this case the compatible vector bundles are the holomorphic ones and the involutive cohomology is the Dolbeault cohomology. The compactly supported cohomology is also the usual compactly supported Dolbeault cohomology.

2. If $T^{0,1} \cap \overline{T^{0,1}} = \{0\}$ then the involutive structure is a *CR-structure*. This is the structure acquired by a real hypersurface in a complex manifold. Our definition is wider than most since it includes "higher codimension" cases and includes also complex manifolds. The involutive cohomology is often known in this case as $\bar{\partial}_b$ cohomology (the "b" standing for boundary).

3. Let $\eta : F \to Z$ be a fibre bundle. Then $\mathcal{C}^{1,0} = \eta^*(\mathcal{E}^1_Z)$ is an involutive structure on F. Bundles on F which are pull-backs of bundles on M are compatible. Suppose the fibres of η have finite-dimensional de Rham cohomology. Then the k-th cohomology of the fibre defines a vector bundle $\mathcal{H}^k \to Z$. The involutive cohomology in this case (often called "relative de Rham cohomology") is fiber-wise de Rham cohomology, parametrised by Z. To be precise, we have
$$H^p_{\mathcal{C}}(F, \eta^* V) = \Gamma(Z, V \otimes \mathcal{H}^p).$$

The principal calculational tool of [BES] concerns the situation where one has two involutive structures $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ on a manifold M. Defining $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$ we also have a short exact sequence
$$0 \to B^1 \to \mathcal{A}^{0,1} \to \mathcal{E}^{0,1} \to 0.$$
Using this to filter the complex for computing $H^*_{\mathcal{A}}(M, V)$ we obtain:

Proposition 2.4. *Let $\mathcal{A}^{1,0} \subset \mathcal{E}^{1,0}$ be two involutive structures on the manifold M and let $V \to M$ be an $\mathcal{A}$-compatible vector bundle. Then V is also $\mathcal{E}$-compatible, and there is a spectral sequence*
$$E_1^{p,q} = H^q_{\mathcal{E}}(M, B^p \otimes V) \Longrightarrow H^{p+q}_{\mathcal{A}}(M, V)$$
where $B^p := \wedge^p B^1$.

Taking instead the compactly supported cohomology the same filtration gives a spectral sequence
$$E_1^{p,q} = H^q_{\mathcal{E},c}(M, B^p \otimes V) \Longrightarrow H^{p+q}_{\mathcal{A},c}(M, V)$$

The proof is straightforward (see [BES] for details) and the compactly supported case is completely analogous.

3 The Penrose transform

We summarise the Penrose transform here as presented in [BES] in a simple case. (See also [BD, T].) The initial data are a double fibration of smooth oriented manifolds
$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & X \end{tikzcd} \tag{1}$$
with the following properties.

1. Z has an involutive structure $\mathcal{Q}$ which is a CR-manifold. (We recall that our definition includes the possibility that Z is complex.)

2. The maps η and τ are fiber-bundle projections such that τ has compact complex fibers.

3. The map η embeds the fibres of τ as holomorphic submanifolds of Z. (A submanifold of a CR-manifold is holomorphic if the involutive structure of Z restricts to give a complex structure on the submanifold.)

In this situation, we can endow F with the involutive structure $\mathcal{E}$ defined by $\mathcal{E}^{1,0}$ being the annihilator of the τ-vertical vectors that are of type $(0,1)$ with respect to the complex structure on the fibres of τ.

Let $V \to Z$ be a $\mathcal{Q}$-compatible vector bundle on Z. The Penrose transform proceeds in three stages.

Step 1 Define an involutive structure $\mathcal{A}$ on F by $\mathcal{A}^{1,0} = \eta^*\mathcal{Q}^{1,0}$. Then $\mathcal{Q}$-cohomology on Z and $\mathcal{A}$-cohomology on F can be related as follows. Introduce first the involutive structure $\mathcal{C}^{1,0} = \eta^*\mathcal{E}^1_Z$ on F. This is exactly the third example in §2. On F we have

$$0 \to \mathcal{A}^{1,0} \to \mathcal{C}^{1,0} \to \eta^*\mathcal{Q}^{0,1} \to 0$$

and the standard spectral sequence of Proposition 2.4 gives

$$E_1^{p,q} = H^q_{\mathcal{C}}(F, \eta^*V \otimes \eta^*\mathcal{Q}^{0,p}) \Longrightarrow H^{p+q}_{\mathcal{A}}(F, \eta^*V).$$

Using the identification of $H_{\mathcal{C}}$ in §2 we obtain

$$E_2^{p,q} = H^p_{\mathcal{Q}}(Z, V \otimes \mathcal{H}^q).$$

A particular case of importance is where the fibres of η are contractible and we deduce that

$$H^p_{\mathcal{Q}}(Z, V) \cong H^p_{\mathcal{A}}(Z, V)$$

with the map being given by pull-back of a representative form.

Step 2 We define $B^1 \to F$ by $B^1 := \mathcal{E}^{1,0}/\mathcal{A}^{1,0}$ and employ the standard spectral sequence of Proposition 2.4 to obtain

$$E_1^{p,q} = H^q_{\mathcal{E}}(F, \eta^*V \otimes B^p) \Longrightarrow H^{p+q}_{\mathcal{A}}(F, \eta^*V).$$

Step 3 We identify the $\mathcal{E}$-cohomology groups appearing in the previous step. The $\mathcal{E}$-cohomology is the Dolbeault cohomology of the fibres of τ parametrised over X. Let us assume that a complex $\mathcal{E}$-compatible vector bundle $E \to F$ is such that the dimension of the Dolbeault cohomology $H^p(\tau^{-1}(x), E|_{\tau^{-1}x})$ is constant as $x \in X$ varies. Then this cohomology defines a vector bundle $\tau_*^p E \to X$. (It is in fact the p-th direct image of the sheaf of smooth sections of E holomorphic on each fibre of τ.) Then

$$H^p_{\mathcal{E}}(F, E) = \Gamma(X, \tau_*^p E).$$

Defining

$$V_{p,q} = \tau_*^q(\eta^* V \otimes B^p)$$

we thus have

$$E_1^{p,q} = \Gamma(X, V_{p,q}) \Longrightarrow H^{p+q}_{\mathcal{A}}(F, \eta^* V).$$

The maps are first order differential operators.

Combining the above steps *in the case where η has contractible fibres* we arrive at the *Penrose transform* which is the spectral sequence

$$E_1^{p,q} = \Gamma(X, V_{p,q}) \Longrightarrow H^{p+q}_{\mathcal{Q}}(Z, V).$$

4 The compactly supported transform

We retain the setting of §3, supposing also that the fibers of η have finite dimensional compactly supported de Rham cohomology. We proceed by analogy with the real Penrose transform, using the fact that the spectral sequence of Proposition 2.4 is valid also for compactly supported cohomology. In the first step we will need to identify the compactly supported cohomology of the involutive structure $\mathcal{C}$ on F.

Lemma 4.1. *Consider the involutive structure $\mathcal{C}^{1,0} = \eta^*(\mathcal{E}^1_Z)$ on F (as in Step 1 for the real Penrose transform). For $V \to Z$ a vector bundle and $k \geq 0$,*

$$H^k_{\mathcal{C},c}(F, \eta^* V) \cong \Gamma_c(Z, \mathcal{H}^k_c \otimes V)$$

where $\mathcal{H}^k_c$ is the bundle whose fiber over $z \in Z$ is the k-compactly supported de Rham cohomology of $\eta^{-1}(z)$.

Proof. We will use only the case where the fibres of η are contractible in this paper, when one can apply fiber by fiber the homotopy formula for compactly supported cohomology (see e.g. [BT, §4]). (The general case then follows from a "Cech de Rham complex" argument.) $\square$

Step 1 Following Step 1 of the Penrose transform in the compactly supported case and using the above Lemma to identify the $\mathcal{C}$-cohomology we obtain the pull-back spectral sequence:

$$E_2^{p,q} = H^p_{\mathcal{Q},\mathrm{c}}(Z, \mathcal{H}^q_c \otimes V) \Longrightarrow H^{p+q}_{\mathcal{A},\mathrm{c}}(F, \eta^* V).$$

If the fibers of η are contractible this immediately converges and we get

$$H^k_{\mathcal{Q},\mathrm{c}}(Z, V) \cong H^{k+d}_{\mathcal{A},\mathrm{c}}(F, \eta^* V)$$

where d is the dimension of the fibres of η. Choose $\rho \in \Gamma(F, \mathcal{C}^{0,d})$ which represents "1" in the compactly supported top-degree cohomology of each fibre. The map is given in this case by pull-back of forms followed by wedging with ρ.

Step 2 We use the standard spectral sequence of Proposition 2.4 for compactly supported cohomology to obtain

$$E_1^{p,q} = H^q_{\mathcal{E},\mathrm{c}}(F, B^p \otimes \eta^* V) \Longrightarrow H^{p+q}_{\mathcal{A},\mathrm{c}}(F, \eta^* V)$$

Step 3 We identify $H^q_{\mathcal{E},\mathrm{c}}(F, \eta^* V \otimes B^p) \cong \Gamma_c(X, V_{p,q})$ where $V_{p,q}$ are exactly the same bundles as arise in the standard transform. This follows almost immediately from the corresponding fact for the usual transform. When the fibres of η are contractible, combining the steps proves the following.

Theorem 4.2. *In the situation of §3 in the case where the fibres of η are contractible there is a spectral sequence*

$$E_1^{p,q} = \Gamma_c(X, V_{p,q}) \Longrightarrow H^{p+q-d}_{\mathcal{Q},c}(Z, V).$$

The vector bundles that appear are exactly those for the standard transform and the differential operators in the spectral sequence are the same as those that appear in the non-compactly supported case, but acting between compactly supported sections of the relevant bundles.

5 The bilinear pairing

For an involutive structure $\mathcal{A}$, define $\kappa_{\mathcal{A}}$ to be the top exterior power of $\mathcal{A}^{1,0}$ (by analogy with the definition of the holomorphic canonical bundle on a complex manifold).

Definition 5.1. *On a manifold F with involutive structure $\mathcal{A}$ and compatible vector bundle V, let $k + l = \operatorname{Rank} \mathcal{A}^{0,1}$. The natural bilinear pairing between involutive cohomology and compactly supported involutive cohomology on F*

$$\int_F : H^k_{\mathcal{A},c}(F, V) \times H^l_{\mathcal{A}}(F, V^* \otimes \kappa_{\mathcal{A}}) \to \mathbb{C},$$

is that given by wedge product of representative forms (combined with contraction between the vector space V and its dual V^*) followed by integration.

When F is a complex manifold and $\mathcal{Q}$ is the complex structure this is the Serre duality pairing that, when F is compact, identifies the (necessarily finite-dimensional) cohomology spaces as mutually dual.

Let F now be the correspondence space for the Penrose transform as previously. We have also on F the corresponding pairing for $\mathcal{E}$-cohomology and a compatible vector bundle E:

$$\int_F : H^q_{\mathcal{E},c}(F, E) \times H^r_{\mathcal{E}}(F, E^* \otimes \kappa_{\mathcal{E}}) \to \mathbb{C}.$$

whenever $q + r = \operatorname{Rank}(\mathcal{E}^{0,1})$ (which is the complex dimension of the fibres of τ).

Lemma 5.2. *Let κ_τ be the line bundle on F which restricts to each fibre of τ to be the canonical bundle of that fibre. (Recall that the fibres of τ are naturally complex manifolds.) Then*

$$\kappa_{\mathcal{E}} = \tau^*(\Lambda_X^{\text{top}}) \otimes \kappa_\tau.$$

(Here Λ_X^{top} denotes the line bundle of complex-valued top-degree forms on X.)

Proof. In the circumstances of the Penrose transform as we have been discussing, there is a short exact sequence of vector bundles on F

$$0 \to \tau^* \mathcal{E}^1_X \to \mathcal{E}^{1,0} \to \mathcal{E}^{1,0}_\tau \to 0$$

where $\mathcal{E}^{1,0}_\tau$ denotes the vector bundle of forms of type $(1, 0)$ in the complex structure of the fibres of τ (so that $\mathcal{E}^{1,0}_\tau = \overline{\mathcal{E}^{0,1}}$). The result follows by taking top exterior powers. $\square$

Proposition 5.3. *Consider the situation in the Penrose transform where we have a $\mathcal{E}$-compatible vector bundle $E \to F$ such that*

$$H^q_{\mathcal{E}}(F, E) = \Gamma(X, \tau^q_* E), \quad H^r_{\mathcal{E},c}(F, E^* \otimes \kappa_{\mathcal{E}}) = \Gamma_c(X, \tau^r_*(E^* \otimes \kappa_{\mathcal{E}})).$$

When $q + r = \operatorname{Rank}(\mathcal{E}^{0,1})$ *we can identify*

$$\tau_*^r(E^* \otimes \kappa_{\mathcal{E}}) = (\tau_*^q E)^* \otimes \Lambda_X^{\mathrm{top}}$$

and the pairing on F *for* $\mathcal{E}$ *cohomology is given by*

$$\int_X : \Gamma_c(X, \tau_*^q E) \times \Gamma(X, (\tau_*^q E)^* \otimes \Lambda_X^{\mathrm{top}}) \to \mathbb{C},$$

where we are contracting the vector bundle $\tau_*^q E$ *with its dual and integrating the resulting top-order form over* X*.*

Proof. Note that by the preceding Lemma,

$$E^* \otimes \kappa_{\mathcal{E}} = E^* \otimes \kappa_\tau \otimes \tau^* \Lambda_X^{\mathrm{top}}$$

and so

$$\tau_*^r(E^* \otimes \kappa_{\mathcal{E}}) = \tau_*^r(E^* \otimes \kappa_\tau) \otimes \Lambda_X^{\mathrm{top}}.$$

For each fibre $\tau^{-1}x$ the cohomology groups

$$H^q(\tau^{-1}x, E) \quad \text{and} \quad H^r(\tau^{-1}x, E^* \otimes \kappa_\tau)$$

are Serre-dual and so we can identify

$$\tau_*^r(E^* \otimes \kappa_{\mathcal{E}}) = (\tau_*^q E)^*.$$

Now split the pairing integral on F into a fibre integral, which is precisely the Serre duality pairing, followed by an integral over the base. $\square$

Proposition 5.4. *Let*

$$E_1^{p,q} = \Gamma_c(X, V_{p,q}) \Longrightarrow H_{\mathcal{A},c}^{p+q}(F, \eta^*(V))$$

and

$$\widetilde{E_1}^{s,t} = \Gamma(X, (V^* \otimes \kappa_{\mathcal{Q}})_{s,t}) \Longrightarrow H_{\mathcal{A}}^{s+t}(F, \eta^*(V^* \otimes \kappa_{\mathcal{A}}))$$

be the spectral sequences for the Penrose transform as discussed above. There is a pairing for $r \geq 1$

$$E_r^{p,q} \times \widetilde{E_1}^{s,t} \to \mathbb{C}, \quad p + s = \operatorname{Rank}(B^1),\ q + t = \operatorname{Rank}(\mathcal{E}^{0,1})$$

which for $r = 1$ *is the pairing for* $\mathcal{E}$*-cohomology on* F *and which converges to the* $\mathcal{A}$*-cohomology pairing on* F*.*

Proof. Note first that $\eta^*\kappa_{\mathcal{Q}} = \kappa_{\mathcal{A}}$. The proof uses the fact that the filtrations of the spectral sequences are induced by a sub-bundle of the $\mathcal{A}^{0,1}$. One can check directly that $\int_F$ descends to the corresponding terms of the r-level of the spectral sequences. $\square$

A similar analysis holds for the pull-back stage of our double fibration where instead of the involutive structure $\mathcal{E}^{1,0}$ we consider the involutive structure $\mathcal{C}^{1,0}$. The outcome in the case where the fibres of η are contractible is that the pairing

$$H^{k-d}_{\mathcal{Q},c}(Z, V) \times H^l_{\mathcal{Q}}(Z, V^* \otimes \kappa_{\mathcal{Q}}) \to \mathbb{C}$$

(where $k + l - d = \operatorname{Rank} \mathcal{Q}^{0,1}$ and d is the dimension of the fibres of τ) pulls back to give the pairing

$$H^k_{\mathcal{A},c}(F, \eta^*(V)) \times H^l_{\mathcal{A}}(F, \eta^*(V^*) \otimes \kappa_{\mathcal{A}}) \to \mathbb{C}.$$

6 Examples

6.1 Euclidean space $\mathbb{R}^3$

Let $X = \mathbb{R}^3$ and let Z be the total space of the holomorphic tangent bundle of $\mathbb{C}P_1$, thought of as the parametrisation space of oriented straight lines in X. One obtains a double fibration where F is the space of "points on oriented lines in $\mathbb{R}^3$".

Let G be the double cover of the group of Euclidean motions of $X = \mathbb{R}^3$. We realise G as

$$G = \{(A, B) \mid A \in SU(2), B \text{ is a } 2 \times 2 \text{ trace-free hermitian matrix}\}$$

acting on $X = \mathbb{R}^3 =$ the space of trace-free Hermitian 2×2 matrices x according to $x \mapsto AxA^* + B$.

As a homogeneous space $Z = G/L$ where

$$L = \left\{ \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \right), \theta \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

For $n \in \mathbb{Z}, \lambda \in \mathbb{C}$ define $\mathcal{O}(n, \lambda)$ to be the G-homogeneous holomorphic line bundle on Z associated to the character $e^{-in\theta - \lambda z}$ of L. Taking $\mathcal{Q}$ to be the complex structure on Z, so that the involutive cohomology is the usual Dolbeault cohomology, it is easy to check that $\kappa_{\mathcal{Q}} = \mathcal{O}(-4, 0)$.

The Penrose transform gives isomorphisms [T]

$$H^1_{\mathcal{Q}}(Z, \mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Ker}(\Delta + 2\lambda^2)$$
$$H^2_{\mathcal{Q}}(Z, \mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Coker}(\Delta + 2\lambda^2)$$

where Δ denotes the Laplacian mapping the space of smooth functions on $\mathbb{R}^3$ to itself.

The fibres of η in this case are 1-dimensional, and so we immediately obtain the following.

Theorem 6.1. *Let Δ_c denote the mapping from the space of compactly supported smooth functions to itself given by the Laplacian. Then*

$$H^0_{\mathcal{Q},c}(Z, \mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Ker}(\Delta_c + 2\lambda^2)$$
$$H^1_{\mathcal{Q}}(Z, \mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Coker}(\Delta_c + 2\lambda^2)$$

(The first observation is trivial since both sides are zero.)

The "Serre duality pairing" on Z

$$H^1_{\mathcal{Q},c}(Z, \mathcal{O}(-2,\lambda)) \times H^1_{\mathcal{Q}}(Z, \mathcal{O}(-2,-\lambda)) \to \mathbb{C}$$

translates into the pairing

$$(f,[g]) \mapsto \int_{\mathbb{R}^3} fg$$

where $f \in \mathrm{Ker}(\Delta + 2\lambda^2)$ and $g \in \mathrm{Coker}(\Delta_c + 2\lambda^2)$.

6.2 A CR example

This is Penrose's original transform, restricted to real Minkowski space (see [W]).

On C^4 with coordinates $z_1, \ldots, z_4$ define the pseudo-hermitian form Φ by

$$\Phi(z,z) = z_1\bar{z}_3 + z_2\bar{z}_4 + z_3\bar{z}_1 + z_4\bar{z}_2.$$

Let I denote the projective line $z_3 = z_4 = 0$ and let Z be the 5-dimensional CR-manifold

$$Z = \{z \in \mathbb{C}P_3 \,|\, \Phi(z,z) = 0\} \setminus I.$$

The space of complex projective lines which lie in Z can be identified with "Minkowski space" $X = \mathbb{R}^4$ with Lorentzian metric.

Taking $\mathcal{Q}$ to be the CR-structure on Z, there are $\mathcal{Q}$-compatible line bundles $\mathcal{O}(n), n \in \mathbb{Z}$ which are the restrictions of the usual holomorphic line bundles on $\mathbb{C}P_3$. We have $\kappa_{\mathcal{Q}} = \mathcal{O}(-4)$. The fibres of η are again 1-dimensional and so we have a "dimension shift" of one for the compactly supported cohomology.

6.2.1 The case of $\mathcal{O}(-2)$

The Penrose transform gives an isomorphism

$$H^1_{\mathcal{Q}}(Z, \mathcal{O}(-2)) \xrightarrow{=} \operatorname{Ker} \square$$

where $\square$ is the wave operator associated to the Lorentz structure on $\mathbb{R}^4$ mapping the space of smooth functions to itself.

We deduce immediately that the compactly supported transform gives isomorphisms

$$H^0_{\mathcal{Q},c}(Z, \mathcal{O}(-2)) \xrightarrow{=} \operatorname{Ker} \square_c$$
$$H^1_{\mathcal{Q},c}(Z, \mathcal{O}(-2)) \xrightarrow{=} \operatorname{Coker} \square_c$$

where $\square_c$ denotes the wave operator mapping the space of compactly supported smooth functions to itself. The first isomorphism is trivial, both sides being zero.

The "Serre duality pairing" on Z

$$H^1_{\mathcal{Q},c}(Z, \mathcal{O}(-2)) \times H^1_{\mathcal{Q}}(Z, \mathcal{O}(-2)) \to \mathbb{C}$$

translates into the pairing

$$(f, [g]) \mapsto \int_{\mathbb{R}^4} fg$$

where $f \in \operatorname{Ker} \square$ and $g \in \operatorname{Coker} \square_c$.

6.2.2 The case of $\mathcal{O}(-1)$ and $\mathcal{O}(-3)$

We will not consider this case in detail. The Penrose transform gives

$$H^1_{\mathcal{Q}}(Z, \mathcal{O}(-3)) \xrightarrow{=} \operatorname{Ker} D^-$$

where D^- is the Dirac operator from smooth sections of the spin bundle S^- to the other spin bundle S^+.

The compactly supported transform gives

$$H^1_{\mathcal{Q},c}(Z, \mathcal{O}(-1)) \xrightarrow{=} \operatorname{Coker} D^+_c$$

where D^+_c is the Dirac operator from compactly supported smooth sections of the spin bundle S^+ to compactly supported smooth sections of S^-.

The Serre duality pairing between these groups becomes the pairing

$$(\alpha, [\beta]) \mapsto \int_{\mathbb{R}^4} \epsilon(\alpha, \beta)$$

where $\alpha \in \operatorname{Ker} D^-$ and $[\beta] \in \operatorname{Coker} D^+$ and ϵ is the complex bilinear skew form on S^-.

6.3 The case of $\mathcal{O}(-4)$ and $\mathcal{O}(0)$

We recall that the smooth complex 2-forms Λ^2 on $X = \mathbb{R}^4$ split as a direct sum of self-dual and anti-self-dual:

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-.$$

The Penrose transform gives

$$H^1_{\mathcal{Q}}(Z, \mathcal{O}(-4)) \xrightarrow{=} \operatorname{Ker}(d : \Lambda^2_- \to \Lambda^3).$$

The Penrose transform on $\mathcal{O}(0)$ gives

$$H^1_{\mathcal{Q}}(Z, \mathcal{O}(0)) \xrightarrow{=} \frac{\operatorname{Ker}(d : \Lambda^1 \to \Lambda^2_-)}{\operatorname{Im} d : \Lambda^0 \to \Lambda^1}$$
$$H^2_{\mathcal{Q}}(Z, \mathcal{O}(0)) \xrightarrow{=} \frac{\Lambda^2_-}{\operatorname{Im} d : \Lambda^1 \to \Lambda^2_-}$$

and so we can immediately deduce that

$$H^0_{\mathcal{Q},c}(Z, \mathcal{O}(0)) \xrightarrow{=} \frac{\operatorname{Ker}(d : \Lambda^1_c \to \Lambda^2_{-,c})}{\operatorname{Im} d : \Lambda^0_c \to \Lambda^1_c}$$
$$H^1_{\mathcal{Q},c}(Z, \mathcal{O}(0)) \xrightarrow{=} \frac{\Lambda^2_{-,c}}{\operatorname{Im} d : \Lambda^1_c \to \Lambda^2_{-,c}}$$

The cohomology group on the left of the first statement is clearly zero and so we deduce that a compactly supported complex-valued 1-form with the self-dual part of its exterior derivative vanishing is necessarily the exterior derivative of a compactly supported function.

The "Serre duality pairing" on Z

$$H^1_{\mathcal{Q},c}(Z, \mathcal{O}(0)) \times H^1_{\mathcal{Q}}(Z, \mathcal{O}(-4)) \to \mathbb{C}$$

translates into the pairing

$$(f, [g]) \mapsto \int_{\mathbb{R}^4} \langle f, g \rangle$$

where f, g are anti-self-dual 2-forms and $\langle f, g \rangle$ is the usual bilinear form.

6.4 Odd-dimensional hyperbolic spaces

In [BD] a twistor correspondence and Penrose transform for $X =$ hyperbolic space of dimension $2n + 1$ is described. It is equivariant with respect to $G =$

$\mathrm{Spin}_0(2n+1,1)$. The space Z is an open orbit in the isotropic Grassmanian of complex n-planes in $\mathbb{C}^{2n+2}$. There are G-homogeneous holomorphic line bundles $\mathcal{O}(n,\lambda), n\in\mathbb{Z}, \lambda\in\mathbb{C}$ on Z.

The smooth Penrose transform gives isomorphisms between Dolbeault cohomologies and kernels and cokernels of operators on H^{2n+1}. (We omit the "$\mathcal{Q}$" for the involutive structure which is the complex structure on Z.)

$$\begin{gathered}
H^{\frac{n(n+1)}{2}}(Z,\mathcal{O}(-2n,\lambda)) \xrightarrow{=} \mathrm{Ker}(\Delta-(\lambda^2-n^2)) \\
H^{\frac{n(n+1)}{2}+1}(Z,\mathcal{O}(-2n,\lambda)) \xrightarrow{=} \mathrm{Coker}(\Delta-(\lambda^2-n^2)) \\
H^{n}(Z,\mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Ker}(\Delta-(\lambda^2-n^2)) \\
H^{n+1}(Z,\mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Coker}(\Delta-(\lambda^2-n^2))
\end{gathered}$$

where Δ denotes the hyperbolic Laplacian defined from the space of smooth functions on H^{2n+1} to itself.

Similar isomorphisms hold for the compactly supported Dolbeault cohomologies, with a shift in dimension by 1 (which is again the dimension of the fibres of η):

$$\begin{gathered}
H_c^{\frac{n(n+1)}{2}-1}(Z,\mathcal{O}(-2n,\lambda)) \xrightarrow{=} \mathrm{Ker}(\Delta_c-(\lambda^2-n^2)) \\
H_c^{\frac{n(n+1)}{2}}(Z,\mathcal{O}(-2n,\lambda)) \xrightarrow{=} \mathrm{Coker}(\Delta_c-(\lambda^2-n^2)) \\
H_c^{n-1}(Z,\mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Ker}(\Delta_c-(\lambda^2-n^2)) \\
H_c^{n}(Z,\mathcal{O}(-2,\lambda)) \xrightarrow{=} \mathrm{Coker}(\Delta_c-(\lambda^2-n^2))
\end{gathered}$$

where Δ_c denotes the hyperbolic Laplacian from the compactly supported smooth functions to itself.

In this case, $\kappa_q = \mathcal{O}(-2n-2,0)$ on Z which has complex dimension $n(n+3)/2$ and so we have Serre-duality pairings

$$\begin{gathered}
H_c^{n}(Z,\mathcal{O}(-2,\lambda)) \times H^{\frac{n(n+1)}{2}}(Z,\mathcal{O}(-2n,-\lambda)) \to \mathbb{C} \\
H_c^{\frac{n(n+1)}{2}}(Z,\mathcal{O}(-2n,\lambda)) \times H^{n}(Z,\mathcal{O}(-2,-\lambda)) \times \to \mathbb{C}
\end{gathered}$$

which both become the pairing

$$\mathrm{Coker}(\Delta_c-(\lambda^2-n^2)) \times \mathrm{Ker}(\Delta-(\lambda^2-n^2)) \to \mathbb{C}$$

induced by multiplication of functions and then integration. The situation with the cohomology giving rise to eigenspinors of the Dirac operator in [BD] is very similar.

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